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Spin structures and Killing spinors on lens spaces

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Abstract. *We determine the values of m and p for which a lens space* $\mathbb{Z}_p \backslash S^{2m-1}$ *admits a spin structure.*
We prove that the only lens spaces (with dimension > 3) admitting a maximal number of linearly independent Killing spinors are the real projective spaces $\mathbb{P}^{4k-1}(\mathbb{R})$.

INTRODUCTION

A lens space is the quotient of the sphere $S^{2m-1} (m \geqslant 2)$ by a particular action of the group $\mathbb{Z}_p$. It is known that there exists a unique spin structure on S^{2m-1} [5]. On the real projective space $\mathbb{P}^{2m-1}(\mathbb{R})$, which is the lens space corresponding to $p = 2$, there exist two inequivalent spin structures when m is even and no spin structure when m is odd [5]. On the lens spaces associated to S^5, there exists one and only one spin structure if p is odd and none if p is even [8].

In this paper, we determine the values of m and p for which a lens space admits a spin structure (Theorem 1).

It was observed in [2] that the spin bundles on $S^n (n \geqslant 2)$ are trivial by

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constructing $2^{[n/2]}$ linearly independent non-zero sections which are Killing spinors. The same argument was used in [4] to prove triviality of the spin bundle on $P^{4k-1}(\mathbb{R})$ $(k \geqslant 1)$. We prove that a lens space of dimension $2m-1$ $(m > 2)$ admits 2^{m-1} linearly independent Killing spinors if and only if $m = 2k$ and the lens space is the projective space $P^{4k-1}(\mathbb{R})$. (Theorem 2).

The paper is organized as follows:
In § 1, we recall the basic notions of spin structure, spinors and Killing spinors. § 2 is devoted to the study of existence and to the construction of spin structures on lens spaces. In § 3, we compute the conditions for a Killing spinor on the sphere to give rise to a Killing spinor on a lens space.

1. DEFINITIONS AND NOTATIONS (for details, see [1] and [2])

Let C_n be the *Clifford algebra* of the real euclidean space of dimension n : $C_n = \mathcal{T}(\mathbb{R}^n)/I$ where $\mathcal{T}(\mathbb{R}^n)$ is the tensor algebra of $\mathbb{R}^n$ and I is the ideal generated by $x \otimes y + y \otimes x + 2\langle x, y\rangle \mathrm{Id}$. ($\langle x, y\rangle$ is the usual scalar product on $\mathbb{R}^n$).
C_n^+ (resp. C_n^-) is the image in C_n of the tensors of even (resp. odd) degree.

If n is even, $n = 2m$, the complexification $C_{2m}^{\mathbb{C}}$ of C_{2m} is isomorphic to the algebra of all linear endomorphisms of the exterior algebra ΛW of an isotropic subspace W of $\mathbb{C}^m$. This isomorphism can be constructed as follows:
Let $\{e_a, a = 1, \ldots, 2m\}$ be an orthonormal basis of $\mathbb{R}^{2m}$.
Let W be the space generated by $\{f_k = e_{2k-1} + ie_{2k},\ 1 \leqslant k \leqslant m\}$
Define

$$\overset{2m}{\widetilde{\rho}} : \mathbb{C}^m (\subset C_{2m}^{\mathbb{C}}) \to \mathrm{End}(\Lambda W) \quad \text{by}$$

$$\overset{2m}{\widetilde{\rho}}(e_{2k-1}) \cdot \alpha = f_k \wedge \alpha - i(f_k^*)\alpha$$

$$\overset{2m}{\widetilde{\rho}}(e_{2k}) \cdot \alpha = -\sqrt{-1}(f_k \wedge \alpha + i(f_k^*)\alpha) \qquad \alpha \in \Lambda W$$

where $i(f_k^*)$ is the inner product by f_k^*.

This linear map extends to an isomorphism of $C_{2m}^{\mathbb{C}}$ onto $\mathrm{End}(\Lambda W)$.

We shall choose as basis of ΛW :

$$\{1, f_I, f_J; f_I = f_{i_1} \wedge \ldots \wedge f_{i_{2r}} \left(1 \leqslant r \leqslant \left[\frac{m}{2}\right],\ 1 \leqslant i_1 \leqslant \ldots < i_{2r} \leqslant m\right),$$

$$f_J = f_{j_1} \wedge \ldots \wedge f_{j_{2r+1}} \left(0 \leqslant r \leqslant \left[\frac{m-1}{2}\right],\ 1 \leqslant j_1 < \ldots < j_{2r+1} \leqslant m\right)$$

The γ *matrices* are the matrices of $\overset{2m}{\gamma}_k = \overset{2m}{\widetilde{\rho}}(e_k)$ in this basis. One has the relations:

$$\overset{2m}{\gamma}_k \overset{2m}{\gamma}_1 + \overset{2m}{\gamma}_1 \overset{2m}{\gamma}_k = -2\delta_{k1} Id.$$

The space $S_{2m} = \Lambda W$ is called the *space of spinors* and has complex dimension 2^m. It decomposes as $S_{2m} = S_{2m}^+ \oplus S_{2m}^-$ where S_{2m}^+ (resp. S_{2m}^-) is the space of even (resp. odd) forms on W. This decomposition is preserved by C^+, i.e. $C^+ S^{\pm} \subset S^{\pm}$ ($\star$).

If n is odd, $n = 2m - 1$, C_{2m-1} is isomorphic to C_{2m}^+ and the isomorphism is constructed as follows:

Let $\alpha : \mathbb{R}^{2m-1}(\subset C_{2m-1}) \to C_{2m}^+ : e_i \to e_i' e_{2m}'$ where $\{e_i, i \leqslant 2m - 1\}$ (resp. $\{e_j', j \leqslant 2m\}$) is an orthonormal basis of $\mathbb{R}^{2m-1}$ (resp. $\mathbb{R}^{2m}$). This extends to an isomorphism of C_{2m-1} onto C_{2m}^+.

Using this isomorphism and ($\star$) one sees that

$$C_{2m-1}^{\mathbb{C}} \sim \mathrm{End}(S_{2m}^+) \oplus \mathrm{End}(S_{2m}^-)$$
$$\equiv \mathrm{End}(S_{2m-1}) \oplus \mathrm{End}(S_{2m-1}').$$

The space $S_{2m}^+ = S_{2m-1}$ is called the space of spinors.

The representation of the Clifford algebra C_{2m-1} on S_{2m-1}, defined on the generators $e_a (a \leqslant 2m - 1)$ by

$$\overset{2m-1}{\widetilde{\rho}}(e_a) = \overset{2m}{\widetilde{\rho}}(\alpha(e_a))|_{S_{2m-1}} = \overset{2m}{\widetilde{\rho}}(e_a' e_{2m}')|_{S_{2m-1}}$$

is irreducible. The γ matrices read $\overset{2m-1}{\gamma}_k = \overset{2m-1}{\widetilde{\rho}}(e_k) = \overset{2m}{\gamma}_k \overset{2m}{\gamma}_{2m}|_{S_{2m}^+}$

The *Spin group*, Spin(n), is the set of elements x in C_n^+ such that $xyx^{-1} \in \mathbb{R}^n(\subset C_n)$ for all $y \in \mathbb{R}^n$ and $x^\tau x = 1$ where τ is the unique antiautomorphism of C_n extending $Id|_{\mathbb{R}^n}$. The fundamental representation of Spin(n) on S_n, $\overset{n}{\widetilde{\rho}}|_{\mathrm{Spin}(n)}$, is called the *spin representation*.

If $n \geqslant 3$, the Spin group Spin(n) is the universal covering of $SO(n)$. The covering homomorphism is $\theta : \mathrm{Spin}(n) \to SO(n)$:

$$x \to [y \to xyx^{-1}].$$

Its differential is an isomorphism of Lie algebras $\theta_\star : \mathrm{spin}(n) \to so(n)$.

If E_{ab} denotes the $n \times n$ matrix with 1 at the intersection of the a^{th} row and b^{th} column and 0 elsewhere, an element Λ of $so(n)$ reads $\Lambda = \Sigma_{a,b} \Lambda^{ab} E_{ab}$ with $\Lambda^{ab} = -\Lambda^{ba}$, and $\theta_\star^{-1}(\Lambda^{ab} E_{ab}) = -\frac{1}{4} \Lambda^{ab} e_a e_b$.

Let (M, g) be an oriented Riemannian manifold of dimension n and let $B \overset{p}{\to} M$

be the bundle of oriented orthonormal frames on M, a principal bundle over M with structure group $SO(n)$. One says that (M, g) admits a *spin structure* (or is a *spin manifold*) if one can find a principal bundle $\widetilde{B} \overset{\widetilde{p}}{\to} M$ over M with structure group $\mathrm{Spin}(n)$ and a homomorphism $\varphi : \widetilde{B} \to B$ such that

(i) the following diagram commutes:

$$\begin{array}{ccc} \widetilde{B} & \overset{\varphi}{\to} & B \\ \widetilde{p}\searrow & & \swarrow p \\ & M & \end{array}$$

(ii) $\varphi(\xi g) = \varphi(\xi)\theta(g)$ for all $\xi \in \widetilde{B}$, $g \in \mathrm{Spin}(n)$, where $\theta : \mathrm{Spin}(n) \to SO(n)$ is the covering homomorphism.

PROPOSITION 1. ([6]): *(M, g) admits a spin structure iff the second Stiefel-Whitney class of the tangent bundle of M vanishes. In this case, the number of inequivalent spin structures on M is equal to $\# H^1(M, \mathbb{Z}_2)$.* ∎

If M is a spin manifold, the associated vector bundle $\mathscr{S} = \widetilde{B} \times_{\overset{n}{\widetilde{\rho}}} S_n$ is called the *spin bundle* and its sections are the *spinor fields.*

To each spinor field ψ, one can associate a function $\widetilde{\psi} : \widetilde{B} \to S_n$ such that

$$\widetilde{\psi}(\xi g) = \overset{n}{\widetilde{\rho}}(g^{-1})\widetilde{\psi}(\xi) \qquad \forall g \in \mathrm{Spin}(n),\ \xi \in \widetilde{B}.$$

Let $\widetilde{\omega}$ be the pull-back connection on $\widetilde{B}$ of the Levi Civita connection ω on $B : \widetilde{\omega} = \theta_\star{}^{-1}(\varphi^\star \omega)$.

The covariant derivative $\nabla\psi$ of a spinor field ψ is defined in the following way: if X is a vector field on M, $\nabla_X \psi$ is the spinor field whose associated function on $\widetilde{B}$ is $\widetilde{\nabla_X \psi} = \overline{X}\widetilde{\psi}$ where $\overline{X}$ is the horizontal lift of X on $\widetilde{B}$ with respect to $\widetilde{\omega}$.

Let $\mathscr{E} = T^\star M \otimes \mathscr{S} \otimes \mathscr{S}^\star = \widetilde{B} \times_{\bar{\rho}} (\mathbb{R}^{n\star} \otimes S_n \otimes S_n^\star)$ where $\bar{\rho} = (\overset{n}{\rho} \circ \theta) \otimes \overset{n}{\widetilde{\rho}} \otimes \overset{n}{\widetilde{\rho}}{}^\star$ (here $\overset{n}{\rho}$ is the usual representation of $SO(n)$ on $\mathbb{R}^n$ and $\overset{n}{\widetilde{\rho}}{}^\star$ is the contragredient representation of $\overset{n}{\widetilde{\rho}}$).

The *element* γ is the section of $\mathscr{E}$ whose associated function is the constant $\widetilde{\gamma} : \widetilde{B} \to \mathbb{R}^{n\star} \otimes S_n \otimes S_n^\star : \widetilde{\gamma}(\xi) = \sum_{k=1}^{n} e_\star^k \otimes \overset{n}{\gamma}_k$ where $\{e_\star^k\}$ is the dual basis of the basis $\{e_k\}$ of $\mathbb{R}^n$, $\overset{n}{\gamma}_k = \overset{n}{\widetilde{\rho}}(e_k)$ and $S_n \otimes S_n^\star$ is identified with $\mathrm{End}(S_n)$.

A *Killing spinor* on (M, g) is a spinor field ψ such that $\nabla\psi = \lambda\gamma\psi$ where λ is a constant. Equivalently, it is a function $\widetilde{\psi} : \widetilde{B} \to S_n$ having the following properties:

(i) $\widetilde{\psi}(\xi g) = \overset{n}{\widetilde{\rho}}(g^{-1})\widetilde{\psi}(\xi) \qquad \forall\, \xi \in \widetilde{B},\ g \in \mathrm{Spin}(n)$

(ii) $(\overline{X}\widetilde{\psi})(\xi) = \lambda \sum_{k=1}^{n} X^k(\xi)\gamma_k \widetilde{\psi}(\xi) \qquad \forall X$ vector field on M where $X^k(\xi)$ $(k \leqslant n)$ are the components of X in the orthonormal frame $\varphi(\xi)$.

A spin manifold admitting a non zero Killing spinor is an Einstein manifold. The constant λ is related to its scalar curvature R by the formula: [2]

$$R = 4n(n-1)\lambda^2$$

2. EXISTENCE OF SPIN STRUCTURES ON LENS SPACES

Consider $S^{2m-1} (m \geqslant 2)$ as the unit sphere of $\mathbb{C}^m : S^{2m-1} = \{(z_1, \dots, z_m) \in \mathbb{C}^m : \sum_{i=1}^{m} z_i\bar{z}_i = 1\}$, and let $\mathbb{Z}_p$ be realized as the subgroup of $U(m)$ (also contained in $SO(2m)$):

$$\left\{ \begin{pmatrix} \exp\left(\frac{2\pi i q_1 k}{p}\right) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & \exp\left(\frac{2\pi i q_m k}{p}\right) \end{pmatrix}, 0 \leqslant k \leqslant p-1 \right\}$$

where $q_j (1 \leqslant j \leqslant m)$ is an integer, $0 < q_j < p$ and q_j is prime to p.

The *lens space* $L(p; q_1, \dots, q_m)$ of dimension $2m-1$ with parameters $p, q_1, \dots, q_m$ is the quotient space $\mathbb{Z}_p \backslash S^{2m-1}$.

Remark. One can take without loss of generality one of the q_j's equal to 1. In what follows, we shall suppose $q_1 = 1$.

Example. The real odd dimensional projective spaces $\mathbb{P}^{2m-1}(\mathbb{R})$ are lens spaces with parameters $(2; 1, \dots, 1)$.

The unique spin structure on S^{2m-1} can be viewed as the principal bundle $\mathrm{Spin}(2m)$ on S^{2m-1} with the homomorphism $\theta : \mathrm{Spin}(2m) \to SO(2m)$ [4]. In fact, $SO(2m)$ is naturally identified with the principal bundle of orthonormal oriented frames on S^{2m-1}: if $e_a (a \leqslant 2m)$ is orthonormal basis of $\mathbb{R}^{2m}$ and if A

is the matrix of an element of $SO(2m)$ in this basis, this element is identified with the orthonormal frame $\{Ae_i, i < 2m\}$ at the point Ae_{2m}.

PROPOSITION 2. *Let (M, g) be a spin manifold of dimension n, and let $(\hat{M}, \hat{g}, p)$ be a Riemannian covering of (M, g). Then $(\hat{M}, \hat{g})$ is a spin manifold. Moreover, if the covering is a Galois covering with automorphism group G and if $\tilde{B} \overset{\varphi}{\to} B$ (resp. $\tilde{\hat{B}} \overset{\hat{\varphi}}{\to} \hat{B}$) is the spin structure on (M, g) (resp. $(\hat{M}, \hat{g})$), then $\hat{B}$ (resp. $\tilde{\hat{B}}$) is a Galois covering of B (resp. $\tilde{B}$) with automorphism group G.*

Proof. The manifold $(\hat{M}, \hat{g})$ is naturally oriented.

There exists an open cover $U_\alpha (\alpha \in A)$ of M which trivializes B, $\tilde{B}$ and $\hat{M}$. The cocycles $g_{\beta\alpha}$ and $\tilde{g}_{\beta\alpha}$ of $\hat{B}$ and $\tilde{\hat{B}}$ are such that $\theta(\tilde{g}_{\beta\alpha}(x)) = g_{\beta\alpha}(x)$ for all x in $U_\beta \cap U_\alpha$.

Let $p^{-1}(U_\alpha) = \coprod_{a \in \mathscr{A}} U_{\alpha,a}$.

The cocycles of $\hat{B}$ and $\tilde{\hat{B}}$ are given by

$$\hat{g}_{\beta_b \alpha_a}(y) = g_{\beta\alpha}(p(y))$$

$$\tilde{\hat{g}}_{\beta_b \alpha_a}(y) = \tilde{g}_{\beta\alpha}(p(y)) \qquad \forall y \in U_{\beta,b} \cap U_{\alpha,a},$$

a and b being such that the intersection is not empty.

We still have $\theta(\tilde{\hat{g}}_{\beta_b \alpha_a}(y)) = \hat{g}_{\beta_b \alpha_a}(y) \qquad \forall y \in U_{\beta,b} \cap U_{\alpha,a}$
and thus we have a spin structure on $\hat{M}$.

For the second part of the proposition, the hypothesis implies that $\hat{M}$ is a principal bundle over M with G as structure group.

So $\hat{M} = \coprod_{\alpha \in A} U_\alpha \times G/\sim$

where $\qquad [x_\alpha, a] \sim [x_\beta, b]$ iff $x_\alpha = x_\beta$ and $b = a \cdot c_{\beta\alpha}(x_\alpha)$

with $c_{\beta\alpha}(x_\alpha) \in G$.

From the preceeding construction of $\hat{B}$, one then has:

$$\hat{B} = \coprod_{\alpha \in A} U_\alpha \times G \times SO(n)/\sim$$

where $\qquad [x_\alpha, a, A] \sim [x_\beta, b, B]$ iff $x_\alpha = x_\beta$, $b = a \cdot c_{\beta\alpha}(x_\alpha)$

and $B = g_{\beta\alpha}(x_\alpha) A$.

So G acts on $\hat{B}$ by left multiplication on the 2^{nd} factor, and this action commutes with the action of $SO(n)$.

A similar argument applied to $\tilde{\hat{B}}$ concludes the proof of the proposition. Q.E.D. ∎

The Riemannian covering $S^{2m-1} \to L(p; q_1, \ldots, q_m)$ is a Galois covering with automorphism group $\mathbb{Z}_p$. Hence Proposition 2 implies that any spin structure on $L(p; q_1, \ldots, q_m)$ is the quotient of Spin $(2m)$ by an action of $\mathbb{Z}_p$. Furthermore, this action is compatible with the homomorphism θ : Spin$(2m) \to SO(2m)$ and hence projects onto the action on $SO(2m)$ of the subgroup $\mathbb{Z}_p$:

$$\mathbb{Z}_p = \left\{ A^k = \exp \frac{2\pi k}{p} \begin{pmatrix} 0 & -q_1 & & & & & \\ q_1 & 0 & & & & & \\ & & 0 & -q_2 & & & \\ & & q_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -q_m \\ & & & & & q_m & 0 \end{pmatrix}, 0 \leqslant k \leqslant p-1 \right\}$$

By the lifting map theorem ([9] th. 1.8.12 p. 36), the action of $\mathbb{Z}_p$ on Spin$(2m)$ is induced by the left action of a subgroup of Spin$(2m)$ isomorphic to $\mathbb{Z}_p$. A generator of this subgroup is one of the 2 elements in $\theta^{-1}(A)$. Hence to detect the existence of a spin structure on $L(p; q_1 \ldots q_m)$, it is sufficient to determine if $\theta^{-1}(A)$ contains an element of order p.

THEOREM 1. *If p is odd, $L(p; q_1, \ldots, q_m)$ admits one and only one spin structure. If p is even, $L(p, q_1, \ldots, q_m)$ doesn't admit a spin structure when m is odd, and admits two inequivalent spin structures when m is even.*

Proof. By the definition of θ_*^{-1}, we have

$$\theta^{-1}(A) = \pm \exp \frac{2\pi}{p} \theta_*^{-1} \begin{pmatrix} 0 & -q_1 & & & \\ q_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -q_m \\ & & & q_m & 0 \end{pmatrix}$$

$$= \pm \exp \frac{\pi}{p} (q_1 e_1 e_2 + q_2 e_3 e_4 + \ldots + q_m e_{2m-1} e_{2m}).$$

Let $\theta^{-1}(A)^+$ and $\theta^{-1}(A)^-$ be these two elements.

In the spin representation $\overset{2m}{\widetilde{\rho}}\,|_{\,\mathrm{Spin}(2m)}$, the element $e_{2j-1}e_{2j}$ is written in the basis $\{1(=f_\phi), f_I = f_{i_1}\wedge\ldots\wedge f_{i_{2r}},\ 1 \leqslant r \leqslant [\frac{m}{2}],\ 1 \leqslant i_1 < \ldots < i_{2r} \leqslant m\}$ of S^+_{2m} as a diagonal matrix with entries:

$$\overset{2m}{\widetilde{\rho}}(e_{2j-1}e_{2j})_{(I,I)} = (\overset{2m}{\gamma}_{2j-1}\overset{2m}{\gamma}_{2j}|_{S^+_{2m}})_{(I,I)} = -i\epsilon^I_j$$

where
$$\epsilon^I_j = \begin{cases} +1 \text{ if } j \in I = \{i_1, \ldots, i_{2r}\} \\ -1 \text{ if } j \notin I = \{i_1, \ldots, i_{2r}\} \end{cases}$$

The matrix of $\theta^{-1}(A)^\pm$ in this basis is thus also diagonal with entries: $(\theta^{-1}(A)^\pm)_{(I,I)} = \pm \exp\frac{-i\pi}{p}(\epsilon^I_1 q_1 + \epsilon^I_2 q_2 + \ldots + \epsilon^I_m q_m)$.

It is of order p iff $(\pm)^p(-1)^{q_1+\ldots+q_m} = 1$.

If p is even, the condition is that $q_1 + \ldots + q_m$ must be even.

But all the q_j's are prime to p, so they are all odd and the condition is satisfied iff m is even.

Note that in this case, the two matrices $\theta^{-1}(A)^+$ and $\theta^{-1}(A)^-$ are of order p.

If p is odd, $\theta^{-1}(A)^+$ is of order p iff $q_1 + \ldots + q_m$ is even, and $\theta^{-1}(A)^-$ is of order p iff $q_1 + \ldots + q_m$ is odd.

In this case, one and only one of the 2 matrices is of order p, and thus there is one and only one spin structure.

When p and m are even, the two spin structures given by $\theta^{-1}(A)^+$ and $\theta^{-1}(A)^-$ are inequivalent. In fact, the universal coefficients theorem ([7]) implies that $\#H^1(L(p; q_1, \ldots, q_m), \mathbb{Z}_2) = 2$ when p is even.

So, applying proposition 1, if there exist spin structures, there are exactly 2 inequivalent ones. The result follows from the fact that all the spin structures on $L(p; q_1, \ldots, q_m)$ are provided by the quotient of Spin$(2m)$ by $\mathbb{Z}_p$ where the generator of $\mathbb{Z}_p$ acts by left multiplication by one of the two matrices $\theta^{-1}(A)^+$ and $\theta^{-1}(A)^-$. Q.E.D. ∎

Applying this result with $p = 2$:

COROLLARY. *For all k integer $\geqslant 1$, $\mathbb{P}^{4k+1}(\mathbb{R})$ does not admit a spin structure and $\mathbb{P}^{4k-1}(\mathbb{R})$ admits two inequivalent spin structures.* ∎

3. KILLING SPINORS ON LENS SPACES

PROPOSITION 3. ([2] - [4]). *On the sphere S^{2m-1} ($m \geqslant 2$), a Killing spinor is defined by a function*

$$\widetilde{\psi}^{\epsilon}(\epsilon = \pm 1) : \mathrm{Spin}(2m) \to S_{2m-1} = S_{2m}^{+}$$

given by $\widetilde{\psi}^{\epsilon}(g) = \tau^{\epsilon}(g^{-1})\psi_e^{\epsilon}$, *where* ψ_e^{ϵ} *is any element of* S_{2m-1} *and* τ^{ϵ} *is the representation of* Spin$(2m)$ *on* S_{2m-1} *whose differential is given by*

$$\tau_{*}^{\epsilon}\left(\sum_{a,b\leqslant 2m} \Lambda^{ab} e_a e_b\right) = \sum_{i,j\leqslant 2m-1} \Lambda^{ij}\gamma_i^{2m-1}\gamma_j^{2m-1} + 2\epsilon \sum_{i\leqslant 2m-1} \Lambda^{i2m}\gamma_i^{2m-1}.$$

It satisfies: $\nabla\psi^{\epsilon} = \frac{\epsilon}{2}\gamma\psi^{\epsilon}$. ∎

Proposition 2 implies that a Killing spinor on a lens space $L(p;\, q_1, \ldots, q_m)$ (which is a spin manifold) admits a lift to S^{2m-1} which is a Killing spinor stable by the action of $\mathbb{Z}_p$.

The Killing spinor ψ^{ϵ} defined by $\widetilde{\psi}^{\epsilon}(g) = \tau^{\epsilon}(g^{-1})\psi_e^{\epsilon}$ on S^{2m-1} gives rise to a Killing spinor on $L(p;\, q_1, \ldots, q_m)$ iff

$$\tau^{\epsilon}(\theta^{-1}(A)^{\pm})\psi_e^{\epsilon} = \psi_e^{\epsilon},$$

where

$$\tau^{\epsilon}(\theta^{-1}(A)^{\pm}) = \tau^{\epsilon}\left(\pm \exp \frac{\pi}{p}\,(q_1 e_1 e_2 + \ldots + q_m e_{2m-1} e_{2m})\right)$$

$$= \pm \exp \frac{\pi}{p}\,(q_1\gamma_1^{2m-1}\gamma_2^{2m-1} + \ldots + q_{m-1}\gamma_{2m-3}^{2m-1}\gamma_{2m-2}^{2m-1} + \epsilon q_m \gamma_{2m-1}^{2m-1}).$$

In the basis $\{1, f_I\}$ of $S_{2m-1} = S_{2m}^{+}$, the matrix of $\tau^{\epsilon}(\theta^{-1}(A)^{\pm})$ is diagonal with entries:

$$(\tau^{\epsilon}(\theta^{-1}(A)^{\pm}))_{(I,I)} = \pm \exp \frac{-i\pi}{p}\,(\epsilon_1^I q_1 + \ldots + \epsilon_{m-1}^I q_{m-1} + \epsilon \cdot \epsilon_m^I q_m)$$

So the equations read:

$$\pm \exp \frac{-i\pi}{p}\,(\epsilon_1^I q_1 + \ldots + \epsilon\epsilon_m^I q_m)(\psi_e^{\epsilon})^I = (\psi_e^{\epsilon})^I$$

where $(\psi_e^{\epsilon})^I$ are the components of ψ_e^{ϵ} in this basis.

There exists a maximal number ($= 2^{m-1}$) of linearly independent Killing spinors iff these equations are simultaneously satisfied for all sets I of an even number of indices.

This is equivalent to

a) $\epsilon_1^I q_1 + \ldots + \epsilon\, \epsilon_m^I q_m = 2k_I p \ \forall I (k_I \in \mathbb{Z})$ if the spin structure is given by $\theta^{-1}(A)^+$

b) $\epsilon_1^I q_1 + \ldots + \epsilon\, \epsilon_m^I q_m = (2k_I + 1)p \ \forall I (k_I \in \mathbb{Z})$ if the spin structures is given by $\theta^{-1}(A)^-$.

The cases a) and b) can be treated in the same way:

Suppose $m > 2$.

By taking $I = \{1, m\}$ and $I = \{2, m\}$ and substracting the 2 equations, we get that $q_1 - q_2$ is a multiple of p. But q_1 and q_2 are positive integers $< p$ and prime to p; so $q_1 - q_2 = 0$ and $q_1 = q_2 = 1$ because we may suppose $q_1 = 1$.

(A similar argument proves that $\forall i, j < m : q_i = q_j = 1$).

By taking $I = \emptyset$ and $I = \{1, 2\}$ and substracting the 2 equations, we obtain that 2 is a multiple of p, so $p = 2$ and all the q_j's (including q_m) are equals to 1. Note that in this case m is even: $m = 2k$.

So this leads necessarily to the case of the real projective space $\mathbb{P}^{4k-1}(\mathbb{R})$. Moreover, the value of ϵ is determined by the spin structure and the parity of k : in the case a) : $\epsilon = (-1)^k$, and in the case b): $\epsilon = (-1)^{k+1}$.

So we have proved:

THEOREM 2. *The only lens spaces $L(p; q_1, \ldots, q_m)$ with $m > 2$ admitting 2^{m-1} linearly independent Killing spinors are such that $m = 2k$ and are the real projective spaces $\mathbb{P}^{4k-1}(\mathbb{R})$ $(k > 1)$.*

Let $\widetilde{B}^{(\pm)}$ denote the two spin structures on $\mathbb{P}^{4k-1}(\mathbb{R})$.

A Killing spinor on $\mathbb{P}^{4k-1}(\mathbb{R})$ with spin structure $\widetilde{B}^{(\pm)}$ is determined by a Killing spinor ψ^ϵ on the sphere S^{4k-1} which satisfies $\widetilde{\psi}^\epsilon(g) = \widetilde{\psi}^\epsilon(\theta^{-1}(A)^{(\pm)}g)$ $\forall g \in$ Spin$(4k)$ and for which the value of ϵ is determined by the condition $\epsilon = (\pm)(-1)^k$. ∎

Remarks. 1. The second part of the theorem was already proved in [4].

2. The first part of the theorem is false when $m = 2$. In this case, there exist two linearly independent Killing spinors on $L(p; q_1, q_2)$ iff $q_1 = q_2 = 1$ for the spin structure $\widetilde{B}^+$ and $q_1 = 1, q_2 = p - 1$ for the spin structure $\widetilde{B}^-$, without conditions about p.

CONCLUDING REMARKS

We have seen that the spin bundle on $\mathbb{P}^{4k-1}(\mathbb{R})(k > 1)$ is trivial. The spheres S^3 and S^7 are the only spheres which admit a trivial principal orthonormal frame bundle ([5] th 13.10 p. 225 and th. 8.2. p. 156). One can show that the spin bundle over any 3-dimensional lens space $L(p; q_1, q_2)$ is trivial. This is not the case for the 7-dimensional lens spaces as was pointed out to us by P. Gilkey [10]. On the other hand, there exist 3-dimensional lens spaces which don't admit Killing spinors. Hence the Killing spinor argument to prove triviality of the spin bundle has limited validity. We need another method to determine whether 3-dimensional lens spaces are the only trivial examples outside projective spaces.

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